**Math 352 Section 1 — Final Exam — Winter 2025**

Mon, Apr 18 – In class

1. Determine all solutions of the equation

$$z^3 - 1 + \sqrt{3}i = 0$$

and express the solutions in *polar* form.

Also, draw a sketch indicating their positions on a circle centered at the origin having the appropriate radius.

*Solution.* If $w$ is a nonzero complex number and $n$ is a positive integer, the solutions of the equation $z^n = w$ are

$$|w|^{1/n} \exp\left(\frac{\mathrm{Arg}(w)i}{n} + \frac{2\pi k i}{n}\right), \qquad k = 0, 1, 2, \ldots, n-1.$$

In our case,

$$z^3 = 1 - \sqrt{3}i = 2\left(\frac{1-\sqrt{3}i}{2}\right) = 2e^{-\pi i/3} = w.$$

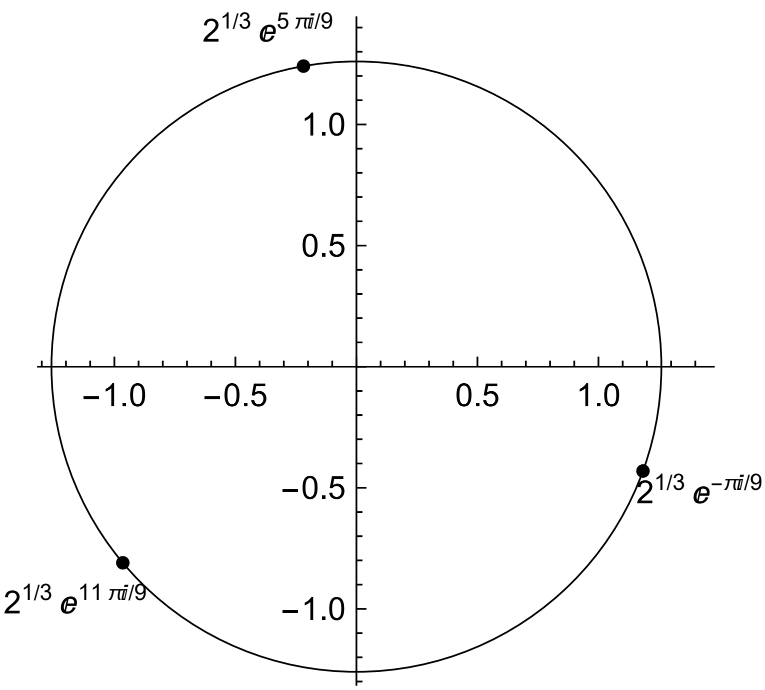
So,

$$\boxed{z = 2^{1/3} e^{-\pi i/9 + 2\pi k i/3}, \quad k = 0, 1, 2}$$

More explicitly, the values of $z$ are

$$\boxed{2^{1/3}e^{-\pi i/9}, \qquad 2^{1/3}e^{5\pi i/9}, \qquad 2^{1/3}e^{11\pi i/9}}$$

Sketch showing the location of the three roots of the equation:




□

2. (a) Determine a real valued function $v(x, y)$ such that

$$f(z) = xy - x + y + iv(x, y)$$

is analytic at all complex $z$. In other words, $v(x, y)$ is a harmonic conjugate of the function $u(x, y) = xy - x + y$.

*Solution.* The functions $u$ and $v$ will need to satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$$\begin{aligned}\frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(xy - x + y) = y - 1 \\ v &= \tfrac{1}{2}y^2 - y + \phi(x)\end{aligned}$$

$$\begin{aligned}\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \\ \phi'(x) &= -\frac{\partial}{\partial y}(xy - x + y) = -x - 1 \\ \phi(x) &= -\tfrac{1}{2}x^2 - x + C\end{aligned}$$

For simplicity, set $C = 0$. Then

$$\boxed{v = \tfrac{1}{2}y^2 - y - \tfrac{1}{2}x^2 - x}$$

□

(b) Express the function $f(z)$ from part (a) in terms of $z$ only. [Note: There should be no $\bar{z}$, $\mathrm{Im}(z)$, or $\mathrm{Re}(z)$ in the formula.]

*Solution.*

$$\begin{aligned}f(z) &= u(x, y) + iv(x, y) \\ &= xy - x + y + i\left(\tfrac{1}{2}y^2 - y - \tfrac{1}{2}x^2 - x\right) \\ &= -\tfrac{i}{2}(x^2 + 2ixy - y^2) - (x + iy) + (-ix + y) \\ &= -\tfrac{i}{2}(x + iy)^2 - (x + iy) - i(x + iy) \\ &= \boxed{-\tfrac{i}{2}z^2 - (1 + i)z}\end{aligned}$$

□

3. Short answers. Fill in the blank with the best answer.

(a) If $f(z)$ is analytic in the punctured disk $0 < |z| < 1$ and if $f(z)$ has a pole of order 3 at $z = 0$, then what is the order of the pole of $f'(z)$ at $z = 0$?

4

(b) After simplification, what is $\mathrm{Log}(e^{3+4i})$?

$3 + (4 - 2\pi)i$

(c) The function $f(z) = \frac{1}{z}$ has an anti-derivative in the annular region $1 < |z| < 2$. Write TRUE or FALSE.

FALSE

(d) If $g(z)$ is an entire function, what is the the value of the integral

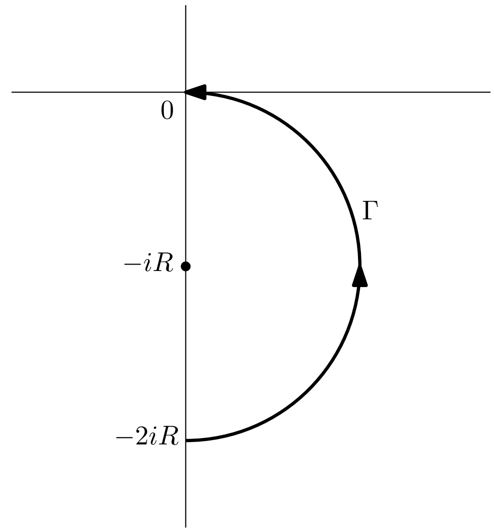
$$\int_{|z-5|=7} \left( \frac{A_{-3}}{z^3} + \frac{A_{-2}}{z^2} + \frac{A_{-1}}{z} + g(z) \right) dz,$$

where the contour is the circle $|z - 5| = 7$ traversed once in the *clockwise* direction.

$-2\pi i A_{-1}$

Grading Note: No partial credit.

4. Compute $\int_\Gamma \bar{z}^2\, dz$ where $\Gamma$ is the semi-circular contour in the diagram. [Note: $\bar{z}^2$ is not analytic. You will need the parametrize the curve.]




*Solution.* Parametrize the contour as

$$\begin{aligned}\Gamma\colon z &= -iR + Re^{i\theta}, \quad -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2} \\ dz &= iRe^{i\theta}.\end{aligned}$$

Then

$$\begin{aligned}\int_\Gamma \bar{z}^2\, dz &= \int_{-\pi/2}^{\pi/2} \left(\overline{-iR + Re^{i\theta}}\right)^2 iRe^{i\theta} d\theta \\ &= iR^3 \int_{-\pi/2}^{\pi/2} (i + e^{-i\theta})^2 e^{i\theta} d\theta \\ &= iR^3 \int_{-\pi/2}^{\pi/2} \left(-1 + 2ie^{-i\theta} + e^{-2i\theta}\right) e^{i\theta} d\theta \\ &= iR^3 \int_{-\pi/2}^{\pi/2} \left(-e^{i\theta} + 2i + e^{-i\theta}\right)\, d\theta \\ &= iR^3 \left[-\tfrac{1}{i}e^{i\theta} + 2i\theta + \tfrac{1}{-i}e^{-i\theta}\right]\Big|_{-\pi/2}^{\pi/2} \\ &= -R^3 \left[e^{i\theta} + 2\theta + e^{-i\theta}\right]\Big|_{-\pi/2}^{\pi/2} \\ &= -R^3 \left([i + \pi - i] - [-i - \pi + i]\right) \\ &= \boxed{-2\pi R^3}\end{aligned}$$

□

5. Let $f(z) = \sum_{k=1}^{\infty} k^2 \left(\frac{z}{3}\right)^k$. Compute each of the following:

(a) $f^{(3)}(0)$

*Solution.*

$$\frac{f^{(3)}(0)}{3!} = 3^2 \cdot \frac{1}{3^3} \quad \Rightarrow \quad f^{(3)}(0) = \frac{3!}{3} = \boxed{2}$$

$\square$

(b) $\displaystyle\oint_{|z|=1} \frac{f(z)}{z^3}\, dz$

*Solution.*

$$\oint_{|z|=1} \frac{f(z)}{z^3}\, dz = \oint_{|z|=1} \frac{1}{z^3} \left( \frac{1^2}{3^1} z + \frac{2^2}{3^2} z^2 + \frac{3^2}{3^3} z^3 + \cdots \right) dz = 2\pi i \cdot \frac{2^2}{3^2} = \boxed{\frac{8\pi i}{9}}$$

$\square$

(c) $\displaystyle\oint_{|z|=1} \cos(z) f(z)\, dz$

*Solution.* $f(z)$ is analytic in the disk $|z| < 3$ and $\cos(z)$ is entire. Their product is analytic everywhere in the disk $|z| < 3$. So,

$$\oint_{|z|=1} \cos(z) f(z)\, dz = \boxed{0}$$

$\square$

(d) $\displaystyle\oint_{|z|=1} \frac{f(z)\sin z}{z^3}\, dz$

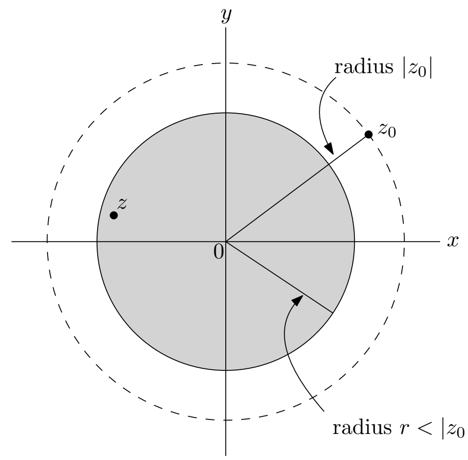
*Solution.*

$$\begin{aligned}
\oint_{|z|=1} \frac{f(z)\sin z}{z^3}\, dz &= \oint_{|z|=1} \frac{1}{z^3} \left( \frac{1^2}{3^1} z + \frac{2^2}{3^2} z^2 + \cdots \right) \left( z - \frac{z^3}{3!} + \cdots \right) dz \\
&= \oint_{|z|=1} \left( \frac{1}{3z} + \cdots \right) dz \\
&= \boxed{\frac{2\pi i}{3}}
\end{aligned}$$

$\square$

6. Assume the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges at the nonzero point $z_0$. Let $0 < r < |z_0|$. Prove that the series converges both *uniformly* and *absolutely* for all $z$ is the smaller closed disk $|z| \le r < |z_0|$.

   Here is a sketch:




*Solution.* Assume the series $\sum_{n=0}^{\infty} a_n z^n$ converges at the point $z_0$. Then

$$\lim_{n\to\infty} a_n z_0^n = 0,$$

which implies that there exists $M > 0$ with

$$|a_n z_0^n| < M$$

for all $n$.

Now let $|z| \le r < |z_0|$. Then

$$|a_n z^n| = |a_n z_0^n| \left|\frac{z}{z_0}\right|^n \le M \left|\frac{r}{z_0}\right|^n \le M \left(\frac{r}{|z_0|}\right)^n.$$

Then

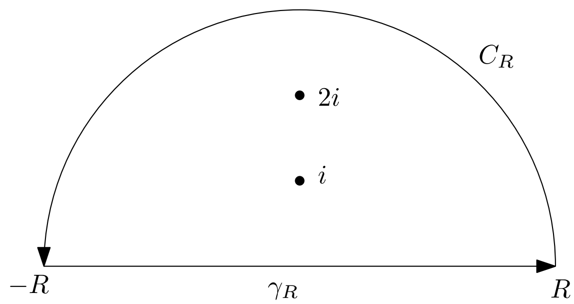
$$\sum_{n=0}^{\infty} |a_n z^n| \le \sum_{n=0}^{\infty} M \left(\frac{r}{|z_0|}\right)^n = \frac{M}{1 - r/|z_0|}.$$

Since the series $\sum_{n=0}^{\infty} |a_n z^n|$ is dominated by a convergent geometric series, by the Weierstrass $M$-test, the series $\sum_{n=0}^{\infty} a_n z^n$ converges both absolutely and uniformly for $|z| \le r < |z_0|$. □

7. Compute the value of the integral

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}.$$

*Solution.* We integrate the function $f(z) = \frac{1}{(z^2+1)(z^2+4)}$ along the contour $\Gamma = \gamma_R + C_R$ in the sketch, where $R > 2$.




Then

$$\begin{aligned}
\int_\Gamma \frac{dz}{(z^2+1)(z^2+4)} &= 2\pi i[\text{Res}(i) + \text{Res}(2i)] \\
&= 2\pi i\left[\frac{1}{(z+i)(z^2+4)}\bigg|_{z=i} + \frac{1}{(z^2+1)(z+2i)}\bigg|_{z=2i}\right] \\
&= 2\pi i\left[\frac{1}{(2i)(-1+4)} + \frac{1}{(-4+1)(4i)}\right] \\
&= \frac{\pi}{3}\left[1 - \frac{1}{2}\right] = \frac{\pi}{6}.
\end{aligned}$$

Letting $R \to \infty$ on the contour $C_R$ gives

$$\left|\int_{C_R} \frac{dz}{(z^2+1)(z^2+4)}\right| \le \frac{\pi R}{(R^2-1)(R^2-1)} \to 0 \quad \text{as} \quad R \to \infty.$$

Letting $R \to \infty$ on the contour $\gamma_R$ gives

$$\int_{\gamma_R} \frac{dz}{(z^2+1)(z^2+4)} \to \text{p.v.}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} \quad \text{as} \quad R \to \infty.$$

This shows that

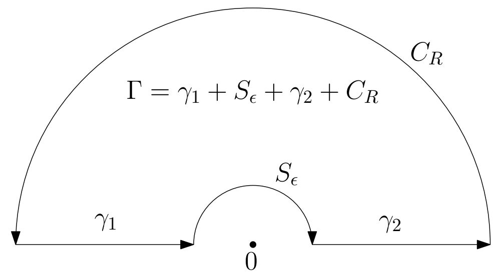
$$\boxed{\text{p.v.}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}}$$

□

8. Compute the value of the integral

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx$$

Hint: Use the contour $\Gamma = \gamma_1 + S_\epsilon + \gamma_2 + C_R$ in the figure.




*Solution.* We will integrate the function $f(z) = \frac{e^{iz}}{z}$ over the contour $\Gamma$. Because $f(z)$ is analytic inside and on $\Gamma$,

$$\int_\Gamma \frac{e^{iz}}{z}\, dz = 0$$

$$\Big(\int_{\gamma_1} + \int_{\gamma_2} + \int_{S_\epsilon} + \int_{C_R}\Big)\frac{e^{iz}}{z}\, dz = 0.$$

By Jordan's lemma,

$$\lim_{R\to\infty}\int_{C_R} \frac{e^{iz}}{z}\, dz = 0.$$

From residue theory,

$$\lim_{\epsilon\to 0}\int_{S_\epsilon} \frac{e^{iz}}{z}\, dz = -\pi i \cdot \text{Res}(0) = -\pi i.$$

Finally,

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}}\Big(\int_{\gamma_1} + \int_{\gamma_2}\Big)\frac{e^{iz}}{z}, dz = \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx$$

Thus,

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx - \pi i + 0 = 0$$

Taking the imaginary part gives

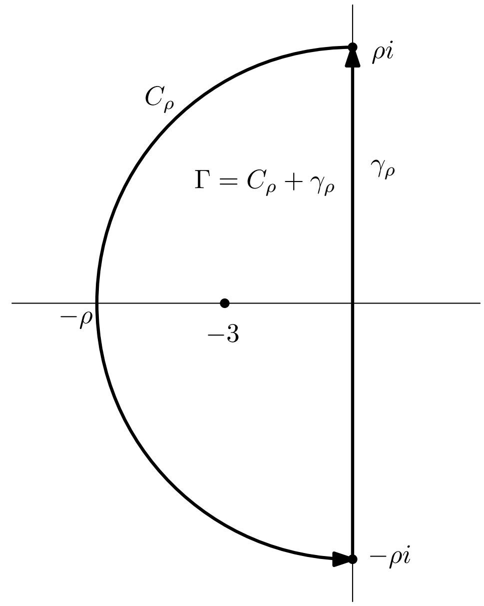
$$\text{p.v.}\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \boxed{\pi}$$ □

9. Prove that the equation

$$z + 3 + 2e^z = 0$$

has precisely one root in the left half-plane.

*Solution.* Consider the semicircular contour $\Gamma$ in the figure.




The function $f(z) = z + 3$ has exactly one root $z = -3$ inside $\Gamma$. Let $h(z) = 2e^z$. We will show that $|h(z)| < |f(z)|$ everywhere on $\Gamma$ so that, by Rouché's theorem, $g(z) = f(z)+h(z) = z + 3 + 2e^z$ will also have exactly one root inside $\Gamma$.

We parametrize the line segment $\gamma_\rho$ by

$$\gamma_\rho\colon z = it, \quad -\rho \leq t \leq \rho.$$

Then on $\gamma_\rho$,

$$|f(z)| = |it + 3| \geq 3 > 2 = |h(z)| = |2e^{it}|.$$

We parametrize the semicircular arc $C_\rho$ (with $\rho > 5$) by

$$C_\rho\colon z = \rho e^{i\theta}, \quad \tfrac{\pi}{2} \leq \theta \leq \tfrac{3\pi}{2}.$$

Then on $C_\rho$,

$$|f(z)| \geq \big||z| - 3\big| = \rho - 3 > 2 \geq |h(z)| = 2e^{\rho\cos\theta} \quad \text{(Note that } \cos\theta \leq 0 \text{ here.)}$$

This shows that $|f(z)| > |h(z)|$ for all $z \in \Gamma$. By Rouché's Theorem, the functions $f(z) = z+3$ and $g(z) = f(z)+h(z) = z+3+2e^z$ have the same number of roots inside $\Gamma$, which is one root. Letting $\rho \to \infty$ gives that $g(z) = z + 3 + 2e^z$ has exactly one root in the left half-plane. $\square$

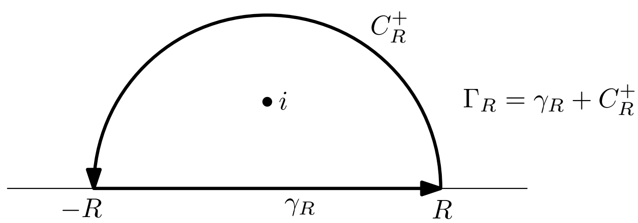
10. Evaluate the integral

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx.$$

*Solution.* We will evaluate the integral

$$\int_{\Gamma_R} f(z)\,dz$$

where $f(z) = \frac{e^{iz}}{z^2+1}$ where $\Gamma_R$ is the contour in the figure.




$$\begin{aligned}\int_{\Gamma_R} \frac{e^{2iz}}{z^2+1}\,dz &= 2\pi i \cdot \operatorname{Res}\left(\frac{e^{iz}}{z^2+1}, i\right)\\ &= 2\pi i \lim_{z\to i}\left((z-i)\frac{e^{iz}}{z^2+1}\right)\\ &= 2\pi i \lim_{z\to i}\frac{e^{iz}}{z+i} = 2\pi i \cdot \frac{e^{-1}}{2i} = \frac{\pi}{e}\end{aligned}$$

By Jordan's Lemma

$$\left|\int_{C_R^+} \frac{e^{iz}}{z^2+1}\right| \to 0 \quad \text{as} \quad R \to \infty.$$

Then

$$\lim_{R\to\infty}\int_{\gamma_R} \frac{e^{iz}}{z^2+1}\,dz = \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx.$$

Thus

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx = \frac{\pi}{e}.$$

Taking the real part gives

$$\boxed{\text{p.v.}\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx = \frac{\pi}{e}}$$

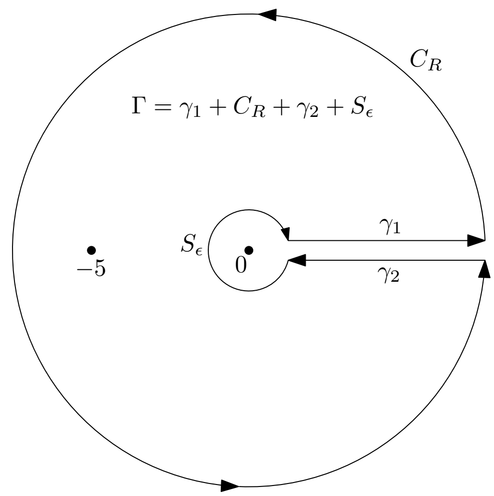
□

11. Evaluate the integral

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+5)},$$

where $\sqrt{x}$ denotes the principal value for $x > 0$.

[Hint: Integrate the function $f(z) = \frac{1}{\sqrt{z}(z+5)}$ around the contour $\Gamma$ in the figure. Use the logarithm $\mathcal{L}_0(z) = \ln|z| + i\arg(z)$, where $0 < \arg(z) \leq 2\pi$, to define the square root $\sqrt{z} = \exp(\frac{1}{2}\mathcal{L}_0(z))$. The contour $\gamma_1$ goes along the 'upper side' of the $x$-axis from $\epsilon$ to $R$ with $\arg(x) = 0$. The contour $\gamma_2$ goes along the 'lower side' of the $x$-axis from $R$ to $\epsilon$ with $\arg(x) = 2\pi$. (The segments $\gamma_1$ and $\gamma_2$ should technically be drawn on the $x$-axis, but are drawn this way to help illustrate the right way to think about the contour.)]




*Proof.* Let $f(z) = \frac{1}{z^{1/2}(z+4)}$ where for $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, we use the branch of $z^{1/2}$ defined by $z^{1/2} = r^{1/2}e^{i\theta/2}$.

On $S_\epsilon$, $z = \epsilon e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

$$\left|\int_{S_\epsilon} f(z)\,dz\right| \leq \frac{2\pi\epsilon}{\epsilon^{1/2}(4-\epsilon)} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

On $C_R$, $z = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$.

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{2\pi R}{R^{1/2}(R-4)} \to 0 \quad \text{as} \quad R \to \infty.$$

On $\gamma_1$, $\arg(x) = 0$. So

$$\int_{\gamma_1} f(z)\,dz = \int_\epsilon^R \frac{dx}{\sqrt{x}(x+4)} \to \int_0^\infty \frac{dx}{\sqrt{x}(x+4)} \quad \text{as } \epsilon \to 0 \text{ and } R \to \infty.$$

On $\gamma_2$, $\arg(x) = 2\pi$. So $x^{1/2} = -\sqrt{x}$, and

$$\int_{\gamma_2} = -\int_{-\gamma_2} f(z)\,dz = -\int_\epsilon^R \frac{dx}{-\sqrt{x}(x+4)} \to \int_0^\infty \frac{dx}{\sqrt{x}(x+4)} \quad \text{as } \epsilon \to 0 \text{ and } R \to \infty.$$

$$\int_\Gamma f(z)\,dz = 2\pi i\,\mathrm{Res}(-5) = 2\pi i \frac{1}{(-5)^{1/2}}\frac{2\pi i}{2i} = 2\pi i \frac{1}{\sqrt{5}i} = \frac{2\pi}{\sqrt{5}}.$$

Letting $\epsilon \to 0$ and $R \to \infty$ in the last expression gives

$$2\int_0^\infty \frac{dx}{\sqrt{x}(x+4)} = \frac{2\pi}{\sqrt{5}}$$

and so

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+4)} = \boxed{\frac{\pi}{\sqrt{5}}}$$

□

12. Using an argument based on Rouché's Theorem, prove that every polynomial of degree $n$ has $n$ zeros.

*Proof.* Let $g(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where $a_n \neq 0$ be a degree $n$ polynomial where $n \geq 1$. Let

$$\begin{aligned} f(z) &= a_n z^n \quad \text{(which has exactly } n \text{ roots)},\\ h(z) &= g(z) - f(z) = a_{n-1}z^{n-1} + \cdots + a_1 z + a_0. \end{aligned}$$

On the circle $|z| = R$,

$$|h(z)| \leq |a_{n-1}|R^{n-1} + \cdots + |a_1|R + |a_0|.$$

We want to choose $R$ large enough such that

$$\underbrace{|a_{n-1}|R^{n-1} + \cdots + |a_1|R + |a_0|}_{\text{upper bound on } |h(z)|} < \underbrace{|a_n|R^n}_{=|f(z)|}$$

$$\frac{|a_{n-1}|}{|a_n|} + \cdots + \frac{|a_1|}{|a_n|}\cdot\frac{1}{R^{n-2}} + \frac{|a_0|}{|a_n|}\cdot\frac{1}{R^{n-1}} < R \qquad (*)$$

Choose $R > 1$ large enough such that

$$\frac{|a_{n-1}|}{|a_n|} + \cdots + \frac{|a_1|}{|a_n|} + \frac{|a_0|}{|a_n|} < R.$$

Then $(*)$ holds. This shows that $|h(z)| < |f(z)|$ on the circle $|z| = R$. So, by Rouché's Theorem, $g(z) = f(z) + h(z)$ has exactly $n$ roots in the disk $|z| < R$. Let $R \to \infty$. Hence, $g(z)$ has exactly $n$ roots in the entire complex plane. □

The End!